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Constrained dynamics and exterior differential systems

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Abstract. The Dirac analysis of constrained Hamiltonian mechanics is one of the conventional precursors to the quantization of classical systems. In this paper the analysis is reformulated in the language of exterior differential systems, starting from the Lagrangian, moving through the generation of primary and secondary constraints and leading to the construction of symmetry generators for gauge symmetries. This reformulation extends the procedure to non-coordinate systems. A computer algebra implementation of the procedure in REDUCE is also described

1. Introduction

Of the various methods of quantization the canonical approach provides a comprehensive and powerful formalism, particularly for constrained systems. Since symmetries play a dominant role in Nature such systems occur frequently and the classical canonical formulation has proved to be a well respected precursor to any attempt at quantization. A systematic procedure for analysing problems based on possibly degenerate Lagrangians has been given by Dirac and his work forms a cornerstone of most modern developments. Rather than remove redundant variables, implied by the symmetries inherent in the description of a problem, Dirac provides an algorithm for constructing a chain of constraints together with a Hamiltonian that describes both the dynamical evolution of the system as well as its gauge freedom. Although in practice the determination of the constraint structure can be a tedious procedure and the classification into first- and second-class constraints a non-trivial problem the method is conceptually straightforward and may be generalized in principle to field systems. Modern developments based on the techniques of symplectic reduction and BRS methods owe much to Dirac's pioneering efforts in this direction. Symmetries of dynamical systems provide much of the data for the standard classical BRS description based on such reduction techniques. Much effort has been spent recently in finding the local symmetry generators corresponding to a set of first-class constraints in Dirac's terminology. There is some subtlety in giving a comprehensive description of such generators and we address this problem in the following.

We have found it useful to reformulate Dirac's procedure for analysing a constrained dynamical system in the language of exterior differential systems. The basic idea is to recognize that the solutions of a (constrained) dynamical problem can be put into correspondence with a class of integral manifolds of a closed involutive exterior

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differential system. Such a reformulation offers one a number of advantages These include the analysis of higher-order problems where the Lagrangian depends on higher-order accelerations, and coordinate independent problems in which the constraints arise naturally from the underlying geometry of the problem. We illustrate this generality with an example from the theory of relativistic elastica. Furthermore, for systems that exhibit a symplectic structure, it is straightforward to generate equations for the local symmetries corresponding to first-class constraints using the notion of an *isovector.*

2. Euler–Lagrange system

The basic setting used here for variational problems in mechanics is as follows. Let X be an n-dimensional manifold, $\{\theta^i\}$ ($i=1,2,\ldots,r$) be a set of 1-forms on X, and let ω be a further 1-form on X, with $\omega\wedge\theta^1\wedge\ldots\wedge\theta^r\neq 0$ at all points of X. The $\{\theta^i\}$ will be called the *constraint system* and ω the *independence form.* Together, they specify a set $\{f\}$ of *integral manifolds*: one-dimensional immersions $f: C\to X$ satisfying

$$f^*\theta^i=0 \tag{2.1}$$

and

$$f^*\omega\neq 0 \tag{2.2}$$

at all points of C.

For example, for first-order variational problems on a configuration space Q with coordinates $\{q^i\}$ and an evolution parameter τ, X would be the jet bundle $J^1(\mathbb{R}\to Q)$ with coordinates $\{\tau, q^i, \dot{q}^i\}$, the constraint system would be the contact system $\{\theta^i = \mathrm{d}q^i-\dot{q}^i\,\mathrm{d}\tau\}$ and the independence form would be $\omega=\mathrm{d}\tau$. The resulting immersions are sections of the jet bundle of the form

$$f:\tau\mapsto\left(\tau, f^i(\tau), \frac{\mathrm{d}f^i}{\mathrm{d}\tau}(\tau)\right) \tag{2.3}$$

As another example, the manifold X for a typical non-coordinate problem might be an orthonormal frame bundle OM over four-dimensional spacetime M, with coframe $\{e^\alpha, \omega^\alpha{}_\beta\}$ ($\alpha,\beta=0,1,2,3$) where $\{e^\alpha\}$ determines an orthonormal coframe on M and $\{\omega^\alpha{}_\beta\}$ are the corresponding connection 1-forms. Here, a constraint system $\{\theta^i\}= \{e^1, e^2, e^3\}$ and an independence form $\omega=e^0$ determine a set of time-like curves in OM which are lifts of curves in M for which the $\{e^\alpha\}$ define a Darboux coframe. A development of this example is treated in section 5.

Within this basic setting, a variational problem is given by specifying a 1-form φ on X and requiring that the action

$$\int_{f(C)}\varphi \tag{2.4}$$

be stationary under variations of the immersion f which preserve the constraint system and independence condition. That is, the problem is to find those f amongst the set determined by $\{\theta^i;\omega\}$ for which the action is stationary. In this paper, we are not concerned with end-point conditions. It has been shown (Griffiths 1982) that the condition that f must satisfy is

$$f^* i_V\,\mathrm{d}\Lambda=0 \qquad \text{for all } V\in T(X) \tag{2.5}$$

where

$$\Lambda = \varphi + p_i \theta^i \tag{2.6}$$

and the p_i are functions on X to be determined.

Following Griffiths, it is useful to extend the manifold X to $Y = X \times \mathbb{R}^r$, introducing the $\{p_i\}$ as r new coordinates rather than functions to be determined during solution. The immersions f lift to immersions into Y, which will continue to be denoted by f. The exterior differential system

$$S = \{ i_V \, d\Lambda | V \in T(Y) \} \tag{2.7}$$

with independence form ω will be called the *Euler–Lagrange system*

Continuing the coordinate example for a first-order variational problem with configuration space Q, suppose the action is specified by a Lagrangian $L(q^i, \dot{q}^i)$. The 1-form φ is then $\varphi = L \, d\tau$ and the Euler–Lagrange system S is

$$S = \begin{cases} dq^i - \dot{q}^i \, d\tau \\ dp_i - L_{q^i} \, d\tau \\ (p_i - L_{\dot{q}^i}) \, d\dot{\tau}. \end{cases} \tag{2.8}$$

Here $L_{q^i} = \partial L / \partial q^i$ and so on. Solving the third of these for the coordinates p_i and substituting into the first gives a system of 1-forms

$$\begin{cases} dq^i - \dot{q}^i \, d\tau \\ dL_{\dot{q}^i} - L_{q^i} \, d\tau \end{cases} \tag{2.9}$$

whose solutions satisfy the usual Euler–Lagrange equations.

There is no difficulty in generalizing to Lagrangians which contain higher-order derivatives of the configuration variables. For a ρth order problem, the first jet bundle is replaced by $J^\rho(\mathbb{R} \to Q)$, with contact system $\{dq^{(\sigma)i} - q^{(\sigma+1)i} \, d\tau\}$, where $\{q^{(\sigma)i}\}$ are jet bundle fibre coordinates, and $\sigma = 0, 1, \ldots, \rho - 1$. The Euler–Lagrange system (2.7) yields the usual Euler–Lagrange equations for higher order problems. For details of this and other examples, see Griffiths (1982), Hartley and Tucker (1990).

3. Constraint analysis

In this section, the coordinate-based example described in section 2 is used to motivate a method of generating the dynamical constraints for a general Euler–Lagrange system.

The key concept is that of *involution* of an exterior differential system on Y with respect to the independence form ω. For the class of problems considered in this paper (where ω is a 1-form), this is the condition that

$$\omega \wedge \alpha \neq 0 \tag{3.1}$$

for all 1-forms α in the exterior system at all points on Y where all 0-forms in the system vanish. This condition singles out those solutions of the exterior differential system for which ω corresponds to an independent e olution variable.

The Euler–Lagrange system (2.7) does not generally satisfy the involution condition. In the case of the exterior system (2.8), the forms

$$(p_i - L_{\dot{q}^i}) \, d\tau \tag{3.2}$$

are clearly not in involution with the independence form $d\tau$.

One approach to solving this problem is to solve the coefficients of $d\tau$ in the offending 1-forms for some of the variables, say the $\{p^i\}$, and eliminate those variables from the rest of the system, pulling back to X with exterior system (2.9). Expanded in terms of the coframe of X, this reads

$$\begin{cases} dq^i - \dot{q}^i \, d\tau \\ L_{\dot{q}^i\dot{q}^j} \, d\dot{q}^j - (L_{q^i} - L_{\dot{q}^i q^j}\dot{q}^j) \, d\tau. \end{cases} \tag{3.3}$$

(It has been assumed here that L does not depend explicitly on τ. Time-dependent Lagrangians can be accommodated by parametrization if required.) If the Lagrangian L is degenerate, so that $\det(L_{\dot{q}^i\dot{q}^j}) = 0$, then this new system on X will not be in involution either, as there will be linear combinations of the second set of forms giving rise to forms proportional to $d\tau$. In this case, the procedure must be repeated, solving for some more variables and pulling back to a smaller space, until an involutive system is found.

For a general exterior differential system, this approach mirrors the process of prolongation of the system to establish a solution submanifold of the Grassmann bundle $G(Y, \omega)$ followed by a projection of this submanifold back down to Y and a pullback of the system onto the resulting submanifold of Y. This procedure is detailed in Griffiths (1982).

Returning to the system (2.8), the conventional approach to finding the corresponding quantum theory, following Dirac (1950), uses a somewhat different analysis. First a Legendre transformation is made to give the Hamiltonian theory, using the conditions

$$p_i - L_{\dot{q}^i} = 0. \tag{3.4}$$

These are solved for as many as possible of the $\dot{q}^i$, which are then eliminated from the Hamiltonian

$$H = p_i \dot{q}^i - L. \tag{3.5}$$

If the Lagrangian is degenerate then some of the conditions (3.4), denoted ϕ_α, will remain after the elimination, dependent on the $\{q^i, p_i\}$ only. These are termed the *primary constraints*. For each primary constraint, one velocity coordinate $\dot{q}^\alpha$ appears explicitly in H. (The primary constraints are not used to simplify H.) Rather than solve the primary constraints for some of the $\{q, p_i\}$, they are appended to the Hamilton equations of motion as further equations, giving an extended set which must be satisfied by any solution:

$$\begin{aligned} \dot{q}^i &= H_{p_i} \\ \dot{p}_i &= -H_{q^i} \\ \phi_\alpha &= 0. \end{aligned} \tag{3.6}$$

(The first set of equations is trivial ($\dot{q}^\alpha = \dot{q}^\alpha$) for those velocities which could not be eliminated.) Consistency of this extended set of equations then requires that the primary constraints be preserved under evolution, which can lead to the generation of further independent equations. Some of these may yield solutions for some of the remaining velocities $\{\dot{q}^\alpha\}$, the rest being new constraints. The new velocity solutions replace trivial ones in (3.6), the new constraints are appended to the extended set of equations, and the procedure is repeated until consistency is achieved.

The analysis in this Dirac procedure is usually simplified (once the primary constraints are obtained) by an adjustment in which the remaining velocities $\{\dot{q}^\alpha\}$ are eliminated in favour of a set of *Dirac multipliers* $\{v^\alpha\}$ in such a way that the Hamiltonian (3.5) takes the form

$$H = H_0 + v^\alpha \phi_\alpha \tag{3.7}$$

where ϕ_α are the primary constraints and $\partial H_0/\partial v^\alpha = 0$. However, it is not necessary to make this change for the purpose of generating the secondary constraints.

In terms of exterior systems, the above procedure may be described as follows. The 1-form Λ on Y with $\theta^i = \mathrm{d}q^i - \dot{q}^i\,\mathrm{d}\tau$ and $\varphi = L\,\mathrm{d}\tau$ can be rewritten using the Hamiltonian (3.5) as

$$\Lambda = -H\,\mathrm{d}\tau + p_i\,\mathrm{d}q^i. \tag{3.8}$$

The Euler–Lagrange system (2.7), (2.8) becomes

$$S = \begin{cases} \mathrm{d}q^i - H_{p_i}\,\mathrm{d}\tau \\ \mathrm{d}p_i + H_{q^i}\,\mathrm{d}\tau \\ H_{\dot{q}^i}\,\mathrm{d}\tau \end{cases} \tag{3.9}$$

Solving as many of the equations $H_{\dot{q}^i} = 0$ as possible for the $\{\dot{q}^i\}$ and pulling back using those solutions leaves a system

$$\begin{cases} \mathrm{d}q^i - H_{p_i}\,\mathrm{d}\tau \\ \mathrm{d}p_i + H_{q^i}\,\mathrm{d}\tau \\ \phi_\alpha\,\mathrm{d}\tau \end{cases} \tag{3.10}$$

in which some of the $\dot{q}^i$ have been eliminated in H_{p_i} and H_{q^i}. This system is clearly not in involution with $\mathrm{d}\tau$ Rather than solve the ϕ_α to overcome the involution problem, they can simply be appended to the exterior system as 0-forms, giving

$$S' = \begin{cases} \phi_\alpha \\ \mathrm{d}q^i - H_{p^i}\,\mathrm{d}\tau \\ \mathrm{d}p_i + H_{q^i}\,\mathrm{d}\tau. \end{cases} \tag{3.11}$$

Since $f^*\phi_\alpha = 0$ implies that $f^*\,\mathrm{d}\phi_\alpha = 0$ also, it is necessary to close the system under exterior differentiation by including the 1-forms $\mathrm{d}\phi_\alpha$ as well, and then re-examine the involution condition. Since the ϕ_α depend solely on the $\{q^i, p_i\}$, $\mathrm{d}\phi_\alpha$ can be rewritten as

$$\mathrm{d}\phi_\alpha = \chi_\alpha\,\mathrm{d}\tau \qquad (\mathrm{mod}\ S'). \tag{3.12}$$

(This means that the expressions in S' have been used to simplify $\mathrm{d}\phi_\alpha$.) If any of the χ_α are independent of the ϕ_α then S' does not give a closed involutive system. If this is the case, then some of the χ_α may yield solutions for some of the remaining $\dot{q}^\alpha$, but the rest are new constraints. The $\dot{q}^\alpha$ solutions may be used to pull back to a smaller space and the new constraints appended as further 0-forms to the system By repeating this procedure until involution is achieved, the Dirac constraint analysis is reproduced.

For non-coordinate problems, it may not be possible to make a distinction between 1-forms corresponding to configuration variables $\{q^i\}$ and those corresponding to velocities $\{\dot{q}^i\}$. To extend the constraint analysis to these problems it is necessary to overcome this difficulty. To do this, it may be noted that it is no more necessary to solve for the $\dot{q}^i$ at any stage of the procedure than it is to solve for the q^i or p_i. It is possible simply to append *all* of the 0-form involution conditions to the system at each step without distinguishing between constraints and velocity solutions. This time, including the exterior derivatives of the 0-forms in the system will lead to some 1-forms containing $d\dot{q}^i$ terms as well as new involution conditions. These new 1-forms are jus. the exterior derivatives of the previous velocity solutions.

Thus the constraint analysis procedure for the general problem is as follows. Start with the Euler–Lagrange system (2.7) and check if it is in involution with the independence form ω. If it is not, then append the involution conditions as 0-forms to the system and include their exterior derivatives. Check this extended system for involution and repeat until an involutive system, closed under exterior differentiation is achieved.

4. Symmetry generators for first class constraints

In the conventional language of the Dirac formalism, the first-class constraints of a Hamiltonian system give rise to the generators of local symmetry transformations. The precise meaning of this statement has been examined in the literature (see, for example, Gràcia and Pons 1988, Gomis *et al* 1990, Henneaux *et al* 1990). For a *local* symmetry transformation, there must be a family of generators parametrized by an arbitrary function of the evolution variable τ. In the language of exterior systems, transformations between solutions of the system are generated by *isovectors* of the system. These are vector fields V which satisfy

$$\mathscr{L}_V S \subseteq S. \tag{4.1}$$

The purpose of this section is to establish the connection between first-class constraints and families of isovectors for exterior differential systems.

According to Dirac, there is a local symmetry corresponding to each primary first-class constraint. This should translate to a family of isovectors of the exterior system for each primary first-class constraint.

It is convenient here to follow the usual practice and transform the unsolved velocities $\{\dot{q}^\alpha\}$ into Dirac multipliers $\{v^\alpha\}$, writing the Hamiltonian as in (3.7).

After carrying out the constraint analysis described in section 3 and eliminating all of the v^α for which solutions become available, the final involutive system contains primary constraints $\{\phi_\alpha\}$ and possibly secondary constraints $\{\chi_\lambda\}$ generated during the analysis. This system resides on some manifold $Y_{\rm inv}$.

Pulled back to $Y_{\rm inv}$, the 1-form Λ still has the form

$$\Lambda = -H\,d\tau + p_i\,dq^i \tag{4.2}$$

so that

$$d\Lambda = -dH \wedge d\tau + \Omega \tag{4.3}$$

where

$$\Omega = dp_i \wedge dq^i \tag{4.4}$$

gives a symplectic form on submanifolds of $Y_{\rm inv}$ where τ is constant and the v^α are given by arbitrarily prescribed functions.

For any function f on $Y_{\rm inv}$, df can be expanded as

$$df = f_\tau\,d\tau + f_{v^\alpha}\,dv^\alpha + \xi_f \tag{4.5}$$

where

$$\xi_f = f_{q^i}\,dq^i + f_{p_i}\,dp_i. \tag{4.6}$$

A corresponding vector field X_f on Y_{inv} can be defined by

$$i_{X_f}\Omega = \xi_f \qquad i_{X_f}\,d\tau = i_{X_f}\,dv^\alpha = 0. \tag{4.7}$$

This gives a Poisson bracket on functions f, g on Y_{inv} defined by

$$[f, g] = i_{X_f} i_{X_g}\Omega = \mathscr{L}_{X_f} g. \tag{4.8}$$

This allows the division of the constraints into first and second class in the usual way. It is not necessary for the present calculation to consider systems with second-class constraints: the considerations for systems with only first-class constraints extend easily to the more general case (for a nice proof of this see Gomis *et al* 1990). So attention below will be restricted to systems with first-class constraints only. These satisfy

$$[f, g] = 0 \qquad (\mathrm{mod}\{\phi_\alpha, \chi_\lambda\}) \qquad \text{for all } f, g \in \{\phi_\alpha, \chi_\lambda\} \tag{4.9}$$

One consequence of this is that no solutions for the Dirac multipliers $\{v^\alpha\}$ arise during the constraint analysis. The involutory system S_{inv} on Y_{inv} can thus be written

$$S_{inv} = \begin{cases} \phi_\alpha \\ \chi_\lambda \\ dq^i - H_{p_i}\,d\tau \\ dp_i + H_{q^i}\,d\tau. \end{cases} \tag{4.10}$$

Its integral manifolds $f: C \to Y_{inv}$ have $\{f^* v^\alpha\}$ as arbitrary functions on C.

Consider a local diffeomorphism generated by a vector field Z of the form

$$Z = A^\alpha \partial_{v^\alpha} + X_G \tag{4.11}$$

for some functions A^α and G on Y_{inv}. (No ∂_τ term has been included in Z because diffeomorphisms which move along the same solution are not of interest here.) In order for Z to be an isovector of S_{inv}, it must satisfy

$$\mathscr{L}_Z g = 0 \qquad (\mathrm{mod}\{\phi_\alpha, \chi_\lambda\}) \tag{4.12}$$

for all $g \in \{\phi_\alpha, \chi_\lambda\}$ and

$$\begin{aligned} \mathscr{L}_Z(dq^i - H_{p_i}\,d\tau) &= 0 \qquad (\mathrm{mod}\ S_{inv}) \\ \mathscr{L}_Z(dp_i + H_{q^i}\,d\tau) &= 0 \qquad (\mathrm{mod}\ S_{inv}). \end{aligned} \tag{4.13}$$

For the first of these,

$$\mathscr{L}_Z g = [G, g] \tag{4.14}$$

since $\partial_{v^\alpha} g = 0$, so (4.12) implies that the function G must be first-class.

The second and third conditions are very similar, and may be dealt with together by combining them to give

$$\mathscr{L}_Z(df - X_f H\,d\tau) = 0 \qquad (\mathrm{mod}\ S_{inv}) \tag{4.15}$$

for all f such that

$$\partial f/\partial\tau = \partial f/\partial v^\alpha = 0. \tag{4.16}$$

For such functions f, $df = \xi_f$, so it follows that

$$\xi_f = X_f H\,d\tau \qquad (\mathrm{mod}\ S_{inv}). \tag{4.17}$$

A short calculation shows that

$$\mathscr{L}_{A^\alpha\partial_{v^\alpha}}(\mathrm{d}f - X_f H\,\mathrm{d}\tau) = -(A^\alpha \partial_{v^\alpha} X_f\,\mathrm{d}(H_0 + v^\beta\phi_\beta))\,\mathrm{d}\tau$$
$$= -(A^\alpha X_f \phi_\alpha)\,\mathrm{d}\tau$$

since by (4.7) ∂_{v^α} and X_f commute. Also,

$$\mathscr{L}_{X_G}(\mathrm{d}f - X_f H\,\mathrm{d}\tau) = \mathrm{d}(X_G f) - (X_G X_f H)\,\mathrm{d}\tau$$
$$= \mathrm{d}[G,f] - (X_f X_G H)\,\mathrm{d}\tau - X_{[G,f]}H\,\mathrm{d}\tau$$
$$= -X_f G_\tau\,\mathrm{d}\tau - X_f G_{v^\alpha}\,\mathrm{d}v^\alpha - X_f X_G H\,\mathrm{d}\tau \qquad (\mathrm{mod}\ S_{\mathrm{inv}})$$

using (4.5), (4.17) and the Jacobi identity Thus

$$\mathscr{L}_Z(\mathrm{d}f - X_f H\,\mathrm{d}\tau) = -\mathscr{L}_{X_f}(G_{v^\alpha}\,\mathrm{d}v^\alpha + (G_\tau + X_G H + A^\alpha\phi_\alpha)\,\mathrm{d}\tau) \qquad (\mathrm{mod}\ S_{\mathrm{inv}}). \tag{4.18}$$

For Z to be an isovector of S_{inv}, the right-hand side of (4.18) must vanish. Suitable functions G and A^α may be constructed from the constraints as follows Relabel the constraints, denoting them collectively by $\{\phi^\rho_\alpha\}$ where $\{\phi^1_\alpha\}$ are the primary constraints $\{\phi_\alpha\}$, $\{\phi^2_\alpha\}$ are the secondary constraints generated from the primaries, and so on. If the constraints have been generated stepwise as in section 3, then there will be a set of relations

$$\mathrm{d}\phi^\rho_\alpha = (\phi^{\rho+1}_\alpha + F^\rho_\alpha(\phi^1_\beta, \ldots, \phi^\rho_\beta))\,\mathrm{d}\tau \qquad (\mathrm{mod}\ S^1_{\mathrm{inv}}) \tag{4.19}$$

where

$$F^\rho_\alpha(\phi^1_\beta, \ldots, \phi^\rho_\beta) = 0 \qquad (\mathrm{mod}\ \phi^1_\beta, \ldots, \phi^\rho_\beta). \tag{4.20}$$

(In (4.19) S^1_{inv} denotes the 1-forms in S_{inv}.) In other words

$$X_{\phi^\rho_\alpha} H = \phi^{\rho+1}_\alpha + F^\rho_\alpha(\phi^1_\beta, \ldots, \phi^\rho_\beta). \tag{4.21}$$

So the constraints at step $\rho+1$ are defined modulo the constraints from steps $1, \ldots, \rho$ only.

If the function G is taken to be

$$G = B^\alpha_\rho \phi^\rho_\alpha \tag{4.22}$$

for some B^α_ρ to be determined, then condition (4.14) is automatically satisfied. In that case, $X_G H$ can be expanded in terms of the constraints using (4.21) as

$$X_G H = (X_{B^\alpha_\rho} H)\phi^\rho_\alpha + B^\alpha_\rho(\phi^{\rho+1}_\alpha + F^\rho_\alpha(\phi^1_\beta, \ldots, \phi^\rho_\beta)). \tag{4.23}$$

Using this result, and with new symbols B^α_0 defined to be zero, the general expression (4.18) reduces to

$$\mathscr{L}_Z(\mathrm{d}f - X_f H\,\mathrm{d}\tau) = -X_f\phi^\rho_\alpha\left(\mathrm{d}B^\alpha_\rho + \left(B^\alpha_{\rho-1} + B^\beta_\sigma \frac{\partial F^\sigma_\beta}{\partial\phi^\rho_\alpha} + A^\alpha\delta^1_\rho\right)\mathrm{d}\tau\right) \qquad (\mathrm{mod}\ S_{\mathrm{inv}}). \tag{4.24}$$

Hence a vector field

$$Z = A^\alpha\partial_{v^\alpha} + X_{B^\alpha_\rho\phi^\rho_\alpha} \tag{4.25}$$

where the functions A^α and B^α_ρ satisfy

$$\mathrm{d}B^\alpha_\rho + \left(B^\alpha_{\rho-1} + B^\beta_\sigma\frac{\partial F^\sigma_\beta}{\partial\phi^\rho_\alpha} + A^\alpha\delta^1_\rho\right)\mathrm{d}\tau = 0 \qquad (\mathrm{mod}\ S_{\mathrm{inv}}) \tag{4.26}$$

is an isovector of the closed involutive system $S_{\rm inv}$ (4.10), and thus generates a symmetry of the equations of motion.

The equations (4.26) are in echelon form: the $B^{\alpha}_{\rho-1}$ are fixed by the B^{α}_{ρ}, and the A^{α} are fixed by the B^{α}_{1}. However, the B^{α}_{ρ} for the maximum value of ρ are completely arbitrary. Since there is one of these arbitrary B^{α}_{ρ} for each primary constraint ϕ^{1}_{α}, it follows that there is an independent isovector Z for each first-class primary constraint, as promised.

In general, the isovector Z obtained from (4.26) will not be independent of the Dirac multipliers v^{α} because of the dependence of the functions F^{ρ}_{α} through (4.21). In fact, to solve (4.26) extra 0 or 1-form conditions must be introduced relating $\mathrm{d}v^{\alpha}$ to $\mathrm{d}p_i$, $\mathrm{d}q^i$ and $\mathrm{d}\tau$, and the resulting B^{α}_{ρ} define an isovector field only for integral manifolds which satisfy these additional conditions. This means that the isovector fields generating symmetry transformations starting from different solution curves will in general be different. To obtain isovector fields which generate symmetry transformations regardless of the starting curve, the functions B^{α}_{ρ}, and A^{α} can be taken to be independent of the v^{α}.

This is achieved by requiring

$$\frac{\partial}{\partial v^{\gamma}}\frac{\partial F^{\sigma}_{\beta}}{\partial \phi^{\rho}_{\alpha}}=0 \qquad (\mathrm{mod}\{\phi^{\rho}_{\alpha}\}) \tag{4.27}$$

which, expanding (4.21), in turn requires that

$$\frac{\partial}{\partial \phi^{\sigma}_{\gamma}}[\phi^{\rho}_{\alpha}, \phi^{1}_{\beta}]=0 \qquad (\mathrm{mod}\{\phi^{\rho}_{\alpha}\}) \tag{4.28}$$

Roughly speaking, this says that $[\phi^{\rho}_{\alpha}, \phi^{1}_{\beta}]=0$ to order $(\phi)^2$. Such a requirement can be satisfied, at least locally, by Abelianizing the constraints (Goursat 1959). However, this will disrupt the hierarchy of constraints, modifying equations (4.19) and (4.21) to give

$$X_{\phi^{\rho}_{\alpha}}H = F^{\rho}_{\alpha}(\phi^{\sigma}_{\beta}) \tag{4.29}$$

where the right-hand side depends upon all of the constraints now. Following this change through gives another set of coupled equations similar to (4.26) except that the echelon structure is lost. Nevertheless, the degree of arbitrariness in the solutions is unchanged: there is one independent isovector for each first-class primary constraint.

Alternatively, if the v^{α} independence requirement is relaxed slightly so that only X_G is required to be independent, not all of Z, then the conditions (4.28) can also be relaxed slightly to

$$\frac{\partial}{\partial \phi^{\sigma}_{\gamma}}[\phi^{\rho}_{\alpha}, \phi^{1}_{\beta}]=0 \qquad (\mathrm{mod}\{\phi^{\rho}_{\alpha}\}) \qquad \sigma=2,3,\ldots \tag{4.30}$$

In this case, the $\mathrm{d}v^{\alpha}$ do not arise in solving equations (4.26), so the resulting isovector field is valid for all solutions. This is the result usually quoted.

5. REDUCE implementation

One of the advantages of the exterior systems approach to the type of problem considered here is its algorithmic nature. However, the calculations can easily involve large expressions and become quite tedious, so this approach is a good candidate for implementation in a computer algebra package. The authors have developed a series of programmes to perform some of the calculations using REDUCE (Hearn 1987), and particularly the EXCALC (Schrüfer 1986) exterior calculus and GROEBNER (Melenk *et al* 1988) Groebner basis packages. These programmes will now be described briefly. Another programme which analyses exterior systems for their solution has been described elsewhere (Hartley and Tucker 1991).

The first procedure is called AMEND. This is a routine for performing the prolongation and projection sequence mentioned in section 2. AMEND is slightly more general than is needed here because it is designed for p-dimensional problems, where the independence form is replaced by a set of p independence 1-forms.

AMEND takes an exterior system of forms, a coframe for the space Y, and a set of independence 1-forms. The Grassmann bundle of p-planes over Y is constructed first, and fibre coordinates $\{l^v\}$ for the patch consisting of planes in involution with the independence forms are generated, as well as the corresponding contact system. A set of equations for the fibre coordinates is obtained by requiring that the p-plane be a solution of the exterior system. These equations determine an algebraic variety in the Grassmann bundle. If these equations can be solved for the l^v alone, then the original exterior system is prolonged by pulling back the contact system of the Grassmann bundle to a submanifold in the algebraic variety. If the variety equations cannot be solved for the l^v alone, then the projection of this variety onto the base space is not onto. This is the signal that the space Y can be reduced by solving those variety equations which remain after all $\{l^v\}$ have been eliminated. The original system is reduced by pulling it back to the smaller space. If reduction is required, then there could be nonlinear equations to be solved, in which case AMEND solves as far as it can, and returns the remaining (unsolved) conditions to be dealt with by hand.

In both prolongation and reduction, it is possible to show (Kuranishi 1957, Griffiths 1982) that the solutions of the amended system give rise to solutions of the original system, and that after a finite number of repetitions of the procedure, an involutive system will be obtained, unless the system has no solutions at all.

If the aim is to produce an involutive system on Y, without moving to another space, then a routine called CONSTRAINTS can be used. Currently, this routine is restricted to the one-dimensional problems discussed earlier. In CONSTRAINTS, the Euler–Lagrange system is brought into involution by using the scheme described at the end of section 3: the system is checked for involution, any 0-form involution conditions discovered are appended to the system, together with their exterior derivatives, and the process is repeated. There is some difficulty in performing the calculation automatically, because involution must be checked modulo the existing 0-forms (3.12). These are generally nonlinear expressions, so the programme uses Groebner bases and a few special manipulations to perform this check. While this allows a wide variety of problems to be tackled, it also imposes some restrictions: if the expressions involve anything other than polynomials or square roots, then some relations may be overlooked unless they are added by hand. This is, of course, no different to the problems encountered in hand calculations. Particular nonlinear problems require specific treatments.

As an example of the application of these programmes, consider the problem of finding a time-like curve in Minkowski spacetime which optimizes the action for a Lagrangian $L(\kappa_1, \kappa_2)$ where κ_1 and κ_2 are the acceleration and torsion of the curve. This generalizes a problem that has attracted some attention in the recent literature. It offers one of the simplest descriptions of a relativistic particle whose dynamics depends on both the intrinsic and extrinsic geometry of an immersion (Dereli *et al* 1990, Nesterenko 1990, Gomis *et al* 1991). The problem may be set up on an extended orthonormal frame bundle $Y = OM \times \mathbb{R}^3$ over four-dimensional spacetime M, with coframe $\{e^\alpha, \omega^\alpha{}_\beta, d\kappa_a\}$ $(\alpha, \beta = 0, 1, 2, 3, a = 1, 2, 3)$ where the $\{e^\alpha\}$ determine an orthonormal coframe on M and the $\{\omega^\alpha{}_\beta\}$ are the corresponding connection 1-forms. The constraint system can be taken as $\{\theta^i\} = \{e^1, e^2, e^3, \omega^0{}_2, \omega^0{}_3, \omega^3{}_1, \omega^0{}_1 - \kappa_1 e^0, \omega^1{}_2 + \kappa_2 e^0, \omega^2{}_3 + \kappa_3 e^0\}$ and the independence form $\omega = e^0$. This determines a set of time-like curves in Y which are lifts of curves in M. Using the structure equations of the frame bundle and the constraint system, it can be seen that the vectors in the frame $\{N_\alpha\}$ dual to $\{e^\alpha\}$ satisfy the Frenet-Serret equations

$$\begin{aligned}
\nabla_{N_0} N_0 &= \kappa_1 N_1 \\
\nabla_{N_0} N_1 &= \kappa_1 N_0 + \kappa_2 N_2 \\
\nabla_{N_0} N_2 &= -\kappa_2 N_1 + \kappa_3 N_3 \\
\nabla_{N_0} N_3 &= -\kappa_3 N_2
\end{aligned} \tag{5.1}$$

where restriction to the time-like curves having unit tangent N_0 is implied. Thus $\{\theta^i; e^0\}$ determines a set of time-like curves for which $\{e^\alpha\}$ defines a Frenet coframe and $\{\kappa_a\}$ are the Frenet curvatures.

In this partially coordinate-free problem, the 1-form Λ is

$$\Lambda = L(\kappa_1, \kappa_2) e^0 + p_i \theta^i \tag{5.2}$$

and the resulting Euler–Lagrange system is

$$\begin{aligned}
&e^1, e^2, e^3, \omega^0{}_2, \omega^0{}_3, \omega^1{}_3 \\
&\omega^0{}_1 - \kappa_1 e^0 \qquad \omega^1{}_2 + \kappa_2 e^0 \qquad \omega^2{}_3 + \kappa_3 e^0 \\
&dp_1 - (\kappa_1 L + p_2 \kappa_2 - \kappa_1(p_7 \kappa_1 - p_8 \kappa_2 - p_9 \kappa_3)) e^0 \\
&dp_2 + (p_1 \kappa_2 - p_3 \kappa_3) e^0 \\
&dp_3 + p_2 \kappa_3 e^0 \\
&dp_4 + (p_2 - p_5 \kappa_3 + p_7 \kappa_2 - p_8 \kappa_1) e^0 \\
&dp_5 - (p_3 + p_4 \kappa_3 - p_6 \kappa_1) e^0 \\
&dp_6 + (p_5 \kappa_1 - p_8 \kappa_3 + p_9 \kappa_2) e^0 \\
&dp_7 + (p_1 - p_4 \kappa_2) e^0 \\
&dp_8 + (p_4 \kappa_1 + p_6 \kappa_3) e^0 \\
&dp_9 - p_6 \kappa_2 e^0 \\
&(L_{\kappa_1} - p_7) e^0 \\
&(L_{\kappa_2} + p_8) e^0 \\
&p_9 e^0.
\end{aligned} \tag{5.3}$$

This exterior system may be written as a set of coupled ordinary differential equations by using proper time τ and writing $e^0 = d\tau$. Note that the equations decouple into disjoint sets, so that the equations for $\{\kappa_a\}$ may be solved first, and their solutions used to construct the Frenet coframe.

Analysing this system with AMEND, assuming $L_{\kappa_a \kappa_b}$ is non-degenerate gives a reduction at the first stage, using

$$p_7 = L_{\kappa_1} \qquad p_8 = -L_{\kappa_2} \qquad p_9 = 0. \tag{5.4}$$

At the second stage, two branches arise from the equation $p_6 \kappa_2 = 0$, and they must be followed separately. First

$$p_6 = 0 \tag{5.5}$$

leads to

$$p_5 \kappa_1 - L_{\kappa_2} \kappa_3 = 0 \tag{5.6}$$

which, when solved for any of the variables, gives a system in involution. On the other branch,

$$\kappa_2 = 0 \tag{5.7}$$

leads to

$$p_1 L_{\kappa_1 \kappa_2} - (p_4 \kappa_1 + p_6 \kappa_3) L_{\kappa_1 \kappa_1} = 0 \tag{5.8}$$

which, when solved for any of the variables, also gives a system in involution (as long as $L_{\kappa_1 \kappa_1} \neq 0$ when $\kappa_2 = 0$).

Processing the same system with CONSTRAINTS gives the same set of equations, in much the same way. A facility has been incorporated in CONSTRAINTS to allow the user to specify certain expressions which will be assumed not to vanish, and which may therefore be cancelled as a factor in any constraint. In this way, it is possible to follow through the separate branches mentioned above First of all, the 'primary constraints' (5.4) are added These generate two 'velocity equations'

$$\begin{aligned} &\Delta\, d\kappa_1 + (p_1 L_{\kappa_2 \kappa_2} - p_4(\kappa_1 L_{\kappa_1 \kappa_2} + \kappa_2 L_{\kappa_2 \kappa_2}))e^0 \\ &\Delta\, d\kappa_2 - (p_1 L_{\kappa_1 \kappa_2} - p_4(\kappa_1 L_{\kappa_1 \kappa_1} + \kappa_2 L_{\kappa_1 \kappa_2}))e^0 \end{aligned} \tag{5.9}$$

where

$$\Delta = \det L_{\kappa_a \kappa_b} \tag{5.10}$$

and the constraint

$$p_6 \kappa_2. \tag{5.11}$$

Taking the first branch using the constraint (5.5) yields (5.6) as a secondary constraint and a further 'velocity equation'

$$L_{\kappa_2 \kappa_2} \Delta\, d\kappa_3 - (p_3 \kappa_1 \Delta + p_4 \kappa_1 \kappa_3 \Delta + p_1 p_5 L_{\kappa_2 \kappa_2} - p_4 p_5 (\kappa_1 L_{\kappa_1 \kappa_2} + \kappa_2 L_{\kappa_2 \kappa_2}))e^0. \tag{5.12}$$

The corresponding results are obtained for the second branch.

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